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Mechanisms for the generation of edge waves over a sloping beach

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Two mechanisms for the generation of standing edge waves over a sloping beach are described using classical linear water-wave theory. The first is an extension of the result of Yih (1984) to a class of localized bottom protrusions on a sloping beach in the presence of a longshore current. The second is a class of longshore surface-pressure distributions over a beach. In both cases it is shown that Ursell-type standing edge-wave modes can be generated in an appropriate frame of reference. Typical curves of the mode shapes are presented and it is shown how in certain circumstances the dominant mode is not the lowest.

1. Introduction

Edge waves are characterized by the fact that they are confined to the shoreline, decaying with distance out to sea. They may be stationary or travelling waves. The first such edge wave was constructed by Stokes (1846) on the basis of linear theory for waves over a uniformly sloping beach. Stokes' edge wave decays exponentially to zero in the seaward direction and the wavelength $\lambda(=2\pi/k)$ in the longshore direction is related to the wave radian frequency ω by

$$\omega^2 = gk\sin\alpha$$

where α is the beach angle.

Viewed from a frame of reference moving with velocity U in the longshore direction, we find that the corresponding Stokes stationary edge wave satisfies

$$k = \frac{g\sin\alpha}{U^2}.$$

Eckart (1951), using linear shallow-water theory, constructed an infinite sequence of edge-wave modes involving products of exponentials and Laguerre polynomials, for which the relation between frequency and longshore wavenumber is

$$\omega^2 = gk(2n+1)\,\alpha, \quad n \text{ an integer}. \tag{1.1}$$

For small beach slopes the first of these agrees with the Stokes edge-wave mode, but it is clear that these shallow-water-theory solutions cannot be valid at large distances from the shoreline in the seaward direction.

Despite the limitations of shallow-water theory it has been used with good effect in showing how edge waves can be generated. For example it has been observed that hurricanes travelling approximately parallel to a nearby coastline sometimes give rise to edge waves. Munk, Snodgrass & Carrier (1956) have shown how a steadily moving pressure distribution on the free surface can model such edge waves, and obtain good agreement with experimental evidence on both periods and amplitudes of the waves.

Greenspan (1956) has extended this idea to consider a more precise physical model in which the pressure disturbance originates at a finite time. In both these papers it is possible to construct solutions for an arbitrary surface-pressure distribution by exploiting the completeness of the Laguerre polynomials, the eigenfunctions corresponding to the edge waves.

A more accurate theory using the full linearized equations, however, shows that the shallow-water model is wrong even near to the shoreline where it predicts an infinite number of edge-wave modes with frequencies and wavenumbers given by (1.1) for each beach angle α. Thus Ursell (1952) has shown that by linear theory, the correct dispersion relation is

$$\omega^2 = gk \sin(2n+1)\alpha, \quad (2n+1)\alpha < \tfrac{1}{2}\pi, \tag{1.2}$$

agreeing with Stokes' solution for $n = 0$, so that for a *given* beach angle, there are in fact only a finite number of bounded edge-wave modes. The eigenfunctions corresponding to the edge waves are sums of exponentials bounded at the shoreline, decaying out to sea as required. In addition to these edge waves there exists a continuous spectrum with $\omega^2 > gk$ which has been considered by Peters (1952) and Roseau (1958). The Ursell edge waves have been generalized to include density stratification by Greenspan (1970). In contrast to the bounded edge waves, the continuous eigenfunctions, which might be used to describe the changes experienced by an oblique incoming wave from infinity, are extremely complicated for general beach angles. Thus a description of the waves produced by say an arbitrary surface-pressure distribution, using full linear theory, would require an expansion in terms of this mixed spectrum and would present considerable technical difficulties. The same would be true for the determination of the stationary edge waves produced in the lee of a small arbitrary bottom protrusion due to a uniform longshore current.

Yih (1984) overcame this difficulty by considering only special bottom protrusions in the presence of a uniform longshore current U. Thus, using the full linearized theory he was able to construct a solution for $\alpha < \frac{1}{4}\pi$ satisfying all the conditions of the problem and which gave rise to the stationary Stokes edge-wave mode in the lee of the protrusion, and which decayed to zero both upstream of the protrusion and out to sea.

In the present work we show how Yih's work can be extended in two ways. First, by enlarging the class of bottom protrusions we show that a solution can be constructed, using the full linearized theory, which gives rise to a sum of stationary Ursell-type edge-wave modes downstream of the protrusion, with wavenumber related to the longshore current U by

$$k = \frac{g \sin(2n+1)\alpha}{U^2}, \tag{1.3}$$

with n an integer, and $(2n+2)\alpha < \frac{1}{2}\pi$. Secondly we show that a similar special class of moving longshore surface-pressure distributions over a uniform sloping beach also gives rise to Ursell-type edge-wave modes.

In the next section we construct a set of Ursell-type edge-wave potentials which will be used in the subsequent sections.

2. Edge-wave potentials

We choose Cartesian coordinates with y vertically upward, x out to sea and z along the undisturbed shoreline. The beach occupies $y\cos\alpha + x\sin\alpha = 0, x \geqslant 0$ and the fluid $y\cos\alpha + x\sin\alpha > 0, y \leqslant 0$. It is convenient to consider a set of coordinates x', y', z obtained from the original set by a clockwise rotation through an angle α. Then

$$x' = x\cos\alpha - y\sin\alpha, \quad y' = y\cos\alpha + x\sin\alpha. \tag{2.1}$$

Consider the functions $U_n(x,y)$ defined by

$$\begin{aligned} U_n(x,y) = {} & A_{0n}\exp[-k(x\cos\alpha - y\sin\alpha)] \\ & + A_{n+1,n}\exp[-k\{x\cos(2n+1)\alpha + y\sin(2n+1)\alpha\}] \\ & + \sum_{m=1}^{n} A_{mn}\{\exp[-k\{x\cos(2m-1)\alpha + y\sin(2m-1)\alpha\}] \\ & \qquad + \exp[-k\{x\cos(2m+1)\alpha - y\sin(2m+1)\alpha\}]\} \end{aligned} \tag{2.2}$$

for $n \geqslant 1$, and where the sum does not appear for $n = 0$.

It is easily verified that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right) U_n(x,y) = 0. \tag{2.3}$$

Since

$$x\cos(2m\pm 1)\alpha \mp y\sin(2m\pm 1)\alpha = x'\cos 2m\alpha \mp y'\sin 2m\alpha \tag{2.4}$$

it is an easy matter to verify that $U_n(x,y)$ satisfies

$$\frac{\partial U_n}{\partial y'} = \sin\alpha\frac{\partial U_n}{\partial x} + \cos\alpha\frac{\partial U_n}{\partial y} = -k\sin(2n+2)\alpha A_{n+1,n}\exp[-kx'\cos(2n+2)\alpha] \quad \text{on } y' = 0, \tag{2.5}$$

whilst

$$KU_n - \frac{\partial U_n}{\partial y} = 0 \quad \text{on } y = 0 \tag{2.6}$$

provided

$$A_{mn} = -\frac{K + k\sin(2m+1)\alpha}{K - k\sin(2m+1)\alpha} A_{m+1,n'} \quad (m = 0, 1, 2, \ldots, n). \tag{2.7}$$

We note that $U_n(x,y) \to 0$ in the fluid region provided $\cos(2n+2)\alpha > 0$ or $(2n+2)\alpha < \frac{1}{2}\pi$.

It is readily confirmed that $U_0(x,y)$ is identical (apart from notation) with the solution given by Yih (1984, equation (11)).

If $A_{n+1,n} = 0$, then we denote $U_n(x,y)$ by $\bar{U}_n(x,y)$ and

$$\frac{\partial \bar{U}_n}{\partial y'} = 0 \quad \text{on } y' = 0, \tag{2.8}$$

whilst

$$K\bar{U}_n - \frac{\partial \bar{U}_n}{\partial y} = (K - k\sin(2n+1)\alpha) A_{nn}\exp[-kx\cos(2n+1)\alpha], \quad y = 0, \tag{2.9}$$

and then (2.7) holds for m up to $n-1$ only. We note that $\bar{U}_n(x,y) \to 0$ in the fluid region provided $\cos(2n+1)\alpha > 0$ or $(2n+1)\alpha < \frac{1}{2}\pi$.

If we require homogeneous conditions to be satisfied on *both* $y = 0$ and $y' = 0$, then we must choose $A_{n+1,n} = 0$ *and* $K = k\sin(2n+1)\alpha$. In this case the functions

$\bar{U}_n(x,y)$ are precisely the Ursell edge-wave modes and it is straightforward to show from (2.7) that

$$A_{mn} = (-1)^m \prod_{r=1}^{m} \frac{\tan(n-r+1)\alpha}{\tan(n+r)} A_{0n} \tag{2.10}$$

as in Ursell (1952).

It is clear that the functions $\bar{U}_n(x,y), U_n(x,y)$ provide a basis for the generation of edge-wave solutions due to either a surface-pressure distribution described by the right-hand side of (2.9) or a bottom disturbance described by the right-hand side of (2.5). We shall show how these functions can be used to solve the two problems described in §1.

In subsequent sections we shall need certain properties of these functions $U_n(x,y)$, $\bar{U}_n(x,y)$. We define

$$\left.\begin{array}{l} K = \dfrac{k^2U^2}{g}, \quad k_m = \dfrac{gs_m}{U^2}, \quad \text{where } s_m = \sin(2m+1)\alpha, \\ c_m = \cos(2m+1)\alpha, \quad s'_m = \sin(2m+2)\alpha, \quad c'_m = \cos(2m+2)\alpha. \end{array}\right\} \tag{2.11}$$

Then (2.7) may be written

$$A_{mn} = -\frac{k+k_m}{k-k_m} A_{m+1,n} \quad (m = 0, 1, \ldots, n), \tag{2.12}$$

which has solution

$$A_{mn} = (-1)^{n-m+1} P_{mn} A_{n+1,n} \quad (m = 0, 1, \ldots, n), \tag{2.13}$$

where
$$P_{mn}(k) = \prod_{r=m}^{n} \frac{k+k_r}{k-k_r}.$$

Now from (2.2)

$$\begin{aligned} U_n(x,0) &= \sum_{m=0}^{n} (A_{m,n} + A_{m+1,n}) \mathrm{e}^{-kxc_m} \\ &= 2A_{n+1,n}(-1)^{n+1} \sum_{m=0}^{n} (-1)^m \mathrm{e}^{-kxc_m} P_{mn}(k) \frac{k_m}{k+k_m}. \end{aligned} \tag{2.14}$$

We may write
$$P_{mn}(k) = \sum_{r=m}^{n} 2k_r \frac{T_{mnr}}{k-k_r} \quad (m \neq n), \tag{2.15}$$

where
$$T_{mnr} = \prod_{\substack{s=m \\ s \neq r}}^{n} \frac{k_r+k_s}{k_r-k_s} = \prod_{\substack{s=m \\ s \neq r}}^{n} \frac{\tan(r+s+1)\alpha}{\tan(r-s)\alpha}. \tag{2.16}$$

If $A_{n+1,n} = 0$ so that $U_n(x,y) \to \bar{U}_n(x,y)$ it can be shown that with $A_{n+1,n} = 1$ in (2.14)

$$\bar{U}_n(x,0) = \{U_{n-1}(x,0) + \mathrm{e}^{-kxc_n}\} A_{nn} \quad (n \neq 0). \tag{2.17}$$

For $n = 0$,
$$U_0(x,0) = -2A_{10} \mathrm{e}^{-kxc_0} \frac{k_0}{k-k_0} \tag{2.18}$$

and
$$\bar{U}_0(x,0) = A_{00} \mathrm{e}^{-kxc_0}. \tag{2.19}$$

3. A longshore uniform current over a small protrusion on the sloping bed

A uniform current in the z-direction over the beach has potential Uz. We assume there exists a small protrusion described by

$$y' = f(x', z) \tag{3.1}$$

which disturbs this uniform flow. Then the perturbed potential $\phi(x, y, z)$ satisfies

$$\nabla^2\phi = 0 \quad \text{in the fluid,} \tag{3.2}$$

$$U^2\frac{\partial^2\phi}{\partial z^2}+g\frac{\partial\phi}{\partial y} = 0, \quad y = 0 \tag{3.3}$$

for steady disturbances, and

$$\frac{\partial\phi}{\partial y'} = U\frac{\partial f}{\partial z} \quad \text{on } y' = 0. \tag{3.4}$$

The free-surface elevation $\eta(x, z)$ is determined from

$$-U\frac{\partial\eta}{\partial z} = \frac{\partial\phi}{\partial y}. \tag{3.5}$$

We consider special forms for $f(x', z)$ given by

$$f(x', z) = \int_0^\infty b(k)\,\mathrm{e}^{-kx'c'_n}\cos kz\,\mathrm{d}k \tag{3.6}$$

for suitable $b(k)$.

Thus $f(x', z)$ is symmetric about $z = 0$, and decays with increasing x' or z. Broadly speaking, the value of c'_n governs the relative rate of decay in the x' and z directions, since

$$\frac{\dfrac{\partial f}{\partial x'}(0,0)}{\dfrac{\partial^2 f}{\partial z^2}(0,0)} = \frac{c'_n\displaystyle\int_0^\infty kb(k)\,\mathrm{d}k}{\displaystyle\int_0^\infty k^2 b(k)\,\mathrm{d}k}. \tag{3.7}$$

Note that c'_n is restricted to positive values so that $(2n+2)\,\alpha < \frac{1}{2}\pi$. Nonetheless for a beach of angle $\alpha = \frac{1}{30}\pi$, say, n can take integer values up to 6, and the difference between the cases $n = 0$, and $n = 6$ produces a factor of 10 in the relative rates of decay of $f(x', z)$ in the x' and z directions. In particular, if we fix the x' dependence, increasing n has the effect of producing a much sharper ridge in the z-direction. More general shapes can also be treated by replacing $\cos kz$ by $\sin kz$ in (3.6) and using combinations of both solutions.

We construct a solution to the problem by writing

$$\phi(x, y, z) = \int_0^\infty b(k)\,U_n(x, y)\sin kz\,\mathrm{d}k. \tag{3.8}$$

Then, using (2.3), (2.5), (2.6) it is easily verified that condition (3.2) is satisfied, as is (3.3), provided we choose

$$K = \frac{k^2U^2}{g}, \tag{3.9}$$

and condition (3.4) with f given by (3.6) is also satisfied provided

$$A_{n+1,n} = \frac{U}{s'_n}. \tag{3.10}$$

The free-surface elevation follows from (3.5), which, on using (2.6) and (3.9), becomes

$$\eta(x,z) = \frac{U}{g}\int_0^\infty kb(k)\,U_n(x,0)\cos kz\,\mathrm{d}k = \frac{U}{g}\operatorname{Re}\int_0^\infty kb(k)\,U_n(x,0)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k. \quad (3.11)$$

Referring to (2.14) we see that the integrand has poles due to the term $P_{mn}(k)$ at $k = k_r$ $(r = 0, \dots, n)$, so that as it stands the integral (3.11) is undefined.

This is a common feature of such problems and arises from solving a steady-state rather than an initial-value problem. There are a number of ways of getting around this problem. Yih (1984) chooses k_r to have a positive imaginary part which tends to zero finally. The essential aim, however, is to ensure that upstream of the disturbance there are no waves. To ensure this we rewrite (3.11) in the form

$$\eta(x,z) = \frac{U}{g}\operatorname{Re}\int_0^\infty kb(k)\,U_n(x,0)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k \quad (3.12)$$

where the path passes *below* the poles at $k = k_r$ $(r = 0, 1, \dots, n)$. Then for $z < 0$ the integral may be deformed along a line in the fourth quadrant where the negative exponential term in z ensures

$$\eta(x,z) \to 0, \quad z \to -\infty.$$

Downstream of the protrusion, where $z > 0$, the dominant behaviour is governed by the contribution to the integral from the residues at $k = k_r$. Typically, after deforming the integral into the first quadrant, where it is negligible for large positive z, we obtain, as $z \to +\infty$,

$$\operatorname{Re}\int_0^\infty \frac{kb(k)\,\mathrm{e}^{\mathrm{i}kz-kxc_m}\,\mathrm{d}k}{(k+k_m)(k-k_r)}\mathrm{d}k \sim \frac{-2\pi k_r b(k_r)\sin k_r z\,\mathrm{e}^{-k_r xc_m}}{k_r+k_m} \quad (3.13)$$

for suitable $b(k)$, and so from (2.14), (2.15), after interchanging the orders of summation, for $z \to +\infty$

$$\eta(x,z) \sim \frac{8\pi}{k_n'}(-1)^n \sum_{r=0}^{n} k_r^2 b(k_r)\sin k_r z \sum_{m=0}^{r} \frac{(-1)^m k_m T_{mnr}\,\mathrm{e}^{-k_r xc_m}}{k_r+k_m} \quad (n \neq 0), \quad (3.14)$$

and (3.10) has been used with $T_{nnn} \equiv (-1)^n$ and we have defined $k_n' \equiv gs_n'/U^2$.

The special case $n = 0$ has to be treated separately and we obtain

$$\eta(x,z) \sim -\frac{4\pi}{k_0'}k_0^2 b(k_0)\,\mathrm{e}^{-k_0 xc_0}\sin k_0 z. \quad (3.15)$$

4. A longshore-moving pressure distribution over a sloping beach

We choose coordinates fixed with the pressure distribution so as to make full use of the results of the preceding section. If the pressure moves with constant speed U in the longshore direction the equations satisfied by the perturbed steady potential $\phi(x,y,z)$ are (Stoker 1957)

$$\nabla^2\phi = 0 \quad \text{in the fluid}, \quad (4.1)$$

$$U^2\frac{\partial^2\phi}{\partial z^2}+g\frac{\partial\phi}{\partial y} = \frac{U}{\rho}\frac{\partial p}{\partial z} \quad \text{on } y = 0, \quad (4.2)$$

$$\frac{\partial\phi}{\partial y'} = 0, \quad y' = 0. \quad (4.3)$$

The surface elevation is determined by

$$-U\frac{\partial\eta}{\partial z} = \frac{\partial\phi}{\partial y}, \quad y = 0. \tag{4.4}$$

We consider special pressure distribution of the form

$$p(x,z) = \int_0^\infty p(k)\,\mathrm{e}^{-kxc_n}\cos kz\,\mathrm{d}k, \tag{4.5}$$

with

$$(2n+1)\alpha < \tfrac{1}{2}\pi,$$

for suitable $p(k)$. The discussion following (3.6) applies to (4.5) also.

We construct a solution in the form

$$\phi(x,y,z) = \int_0^\infty p(k)\,\bar{U}_n(x,y)\sin kz\,\mathrm{d}k \tag{4.6}$$

which satisfies (4.1), (4.3), and also (4.2) provided

$$K = \frac{k^2U^2}{g}$$

and

$$A_{nn} = \{U\rho(k-k_n)\}^{-1}. \tag{4.7}$$

The surface elevation is derived from (4.4) which, after using (2.9), may be written

$$\eta(x,z) = \eta'(x,z) - \frac{1}{\rho g}p(x,z), \tag{4.8}$$

where

$$\eta'(x,z) = \mathrm{Re}\frac{U}{g}\int_0^\infty kp(k)\,\bar{U}_n(x,0)\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k$$

$$= \mathrm{Re}\frac{1}{\rho g}\int_0^\infty \frac{kp(k)}{k-k_n}\{U_{n-1}(x,0)+\mathrm{e}^{-kxc_n}\}\,\mathrm{e}^{\mathrm{i}kz}\,\mathrm{d}k \tag{4.9}$$

from (2.17) and (4.7).

The subsequent calculation of the contribution to the integral for $z \to \infty$ is facilitated by noting that

$$\frac{P_{m,n-1}}{k-k_n} = \frac{P_{mn}}{k+k_n}. \tag{4.10}$$

Thus we find, in a similar manner to the previous problem in §3 that

$$\eta'(x,z) \sim \frac{8\pi(-1)^n}{\rho g}\sum_{r=0}^{n}\frac{k_r^2\,p(k_r)}{k_r+k_n}\sin k_r z\sum_{m=0}^{r}\frac{(-1)^m k_m\,\mathrm{e}^{-k_r x c_m}}{k_r+k_m}T_{mnr}, \quad z\to\infty, \tag{4.11}$$

provided we define $T_{nnn} \equiv 1$.

The case $n = 0$ is best considered separately whence we find

$$\eta'(x,z) \sim \frac{2\pi}{\rho g}k_0\,p(k_0)\,\mathrm{e}^{-k_0c_0}\sin k_0 z, \quad z\to\infty. \tag{4.12}$$

For example with $p(k) = \mathrm{e}^{-ak}$, we see that the localized pressure

$$p(x,z) = \frac{a+c_0x}{(a+c_0x)^2+z^2} \tag{4.13}$$

produces a train of Stokes standing edge waves of wavelength $2\pi k_0^{-1} = 2\pi U^2/g \sin\alpha$ and of amplitude ratio, defined to be the ratio of the amplitude of the edge wave at the shoreline to $p_{\max}/\rho g$, of $2\pi k_0 a\, \mathrm{e}^{-k_0 a}$.

5. Possible extensions

It is possible to generalize the forcing mechanisms by replacing $\cos kz$ with $\sin kz$ in (3.6) and (4.5) and by using combinations of both. Again the transient development of edge waves can be considered for the same spatial variation (3.6). For example for a transient motion of the beach having variation $a(t)$ with time, the resulting elevation is

$$\eta(x,z,t) = \frac{\mathrm{i}}{g}\int_{-\infty}^{\infty} \omega\bar{a}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\int_0^{\infty} b(k)\,U_n(x,y)\,\cos kz\,\mathrm{d}k\,\mathrm{d}\omega, \tag{5.1}$$

where in the definition of $U_n(x,y), K = \omega^2/g$ and

$$A_{n+1,n} = -(ks_n')^{-1}. \tag{5.2}$$

Here
$$\bar{a}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t. \tag{5.3}$$

For example for $n = 0$ and with $\bar{a}(\omega) = 1$, corresponding to an impulsive motion, we find

$$\eta(x,z,t) = -\sec\alpha\int_0^{\infty} b(k)\,\mathrm{e}^{-kx\cos\alpha}\cos kz\,\cos\omega_0 t\,\mathrm{d}k, \tag{5.4}$$

where
$$\omega_0^2 = gk\sin\alpha.$$

The integral (5.4) is identical with that obtained in the classical Cauchy–Poisson problem in deep water (Lamb 1932, §§238–241) apart from the factor $\sin\alpha$ in the dispersion relation, and can be treated similarly.

6. Discussion of results

The main results of this paper are given by (3.14) and (3.15) which give the surface elevation of standing edge waves downstream of the special bottom protrusion described by (3.6) in the presence of a uniform longshore current U. Similarly, (4.12) and (4.13) describe the surface elevation of standing edge waves downstream of a special pressure distribution described by (4.5) moving with constant speed U in the longshore direction, as viewed from axes moving with the pressure distribution.

Since the form of the solution in each case is so similar we shall concentrate our discussions on the case of the longshore current over a bottom protrusion.

In order to obtain details of the standing edge-wave modes in (3.14) we need to specify the particular form of $b(k)$ and hence the bottom protrusion $f(x',z)$. However some comments can still be made for arbitrary $b(k)$.

For a given bottom slope α, there exists a largest integer n satisfying $\cos(2n+2)\alpha > 0$. For example if $\alpha = \frac{1}{30}\pi$, we can choose $n = 0, 1, \ldots, 6$, and any one of these permissible values of n may be used in (3.6) as ingredients in constructing our bottom protrusion together with $b(k)$. It follows that no edge-wave modes can be constructed by this method if $\cos 2\alpha < 0$ or $\alpha > \frac{1}{4}\pi$.

The solution (3.14) shows us that for a given choice of n the wave elevation downstream of the bottom protrusion is given by a sum of $n+1$ standing edge-wave modes of wavelengths

$$\lambda_r \equiv 2\pi k_r^{-1} = \frac{2\pi U^2}{g\sin(2r+1)\alpha} \quad (r = 0, 1, \dots, n), \tag{6.1}$$

in the longshore direction.

Each edge-wave mode with wavenumber k_r decays with distance out to sea, the x-dependence being a well-defined but complicated sum of exponentials of the form $\exp(-k_r c_m x)$ for $m = 0, 1, \dots, r$. The lowest edge-wave mode corresponding to $r = 0$ (and the only one if $n = 0$) is just the Stokes standing edge-wave mode, decaying monotonically to zero as $x \to \infty$.

It is of interest to compare the amplitude of the lowest edge-wave mode ($r = 0$) in the case of general n to that in the $n = 0$ case. To do this we pick out the coefficient of $\sin k_0 z$ in (3.14) in the two cases of general n and $n = 0$ and find their ratio. This turns out to be

$$\left(\frac{\cos\alpha}{\cos(n+1)\alpha}\right)^2, \tag{6.2}$$

where the identity

$$T_{0n0} = -\frac{\tan(n+1)\alpha}{\tan\alpha} \quad (n \neq 0) \tag{6.3}$$

has been used. For $\alpha = \frac{1}{30}\pi$, $n = 6$ the expression (6.2) is about 1.9. However, it is not clear that the lowest mode is the most dominant anyway for general n since the actual amplitude depends upon $b(k_r)$ and hence upon the particular bottom geometry.

We shall restrict attention to the bottom protrusion derived from choosing

$$b(k) = A k^p \,\mathrm{e}^{-ka}, \quad A, p, a > 0, \tag{6.4}$$

namely,

$$f(x', z) = \frac{Ap!}{\{(a+x'c_n')^2 + z^2\}^{p+1}} \operatorname{Re}(a + x'c_n' - \mathrm{i}z)^{p+1}. \tag{6.5}$$

Then, from (3.14), far downstream of the protrusion the elevation of the standing edge-wave modes is

$$\eta(x,z) \sim \frac{8\pi}{k_n'}(-1)^n \sum_{r=0}^{n} B_r(x)\sin k_r z, \tag{6.6}$$

where

$$B_r(x) = k_r^{p+2}\,\mathrm{e}^{-k_r a} \sum_{m=0}^{r} \frac{(-1)^m s_m}{s_r + s_m} T_{mnr}\,\mathrm{e}^{-k_r c_m x}. \tag{6.7}$$

It is convenient to introduce an amplitude ratio A_r being defined as the ratio of the amplitude at the shoreline of the lowest edge-wave mode ($r = 0$) to the maximum height of the protrusion above the (x', z)-plane at the shoreline, which is just

$$\int_0^\infty k^p\,\mathrm{e}^{-ka}\,\mathrm{d}k = \frac{p!}{a^{p+1}}.$$

Then

$$A_\mathrm{r} = \frac{2\pi}{p!}\frac{(k_0 a)^{p+1}\,\mathrm{e}^{-k_0 a}\cos\alpha}{\cos^2(n+1)\alpha}, \tag{6.8}$$

which reduces to the result obtained by Yih (1984, equations (29) and (33)) for the case $n = 0$, $p = 0$.

The effect of varying a is to vary both the peak of the protrusion at the shoreline and its extent in the longshore direction. As a increases, for fixed α, A_r reaches a maximum at $k_0 a = (p+1)^{-1}$ before tending to zero as $a \to \infty$. Thus, as confirmed by

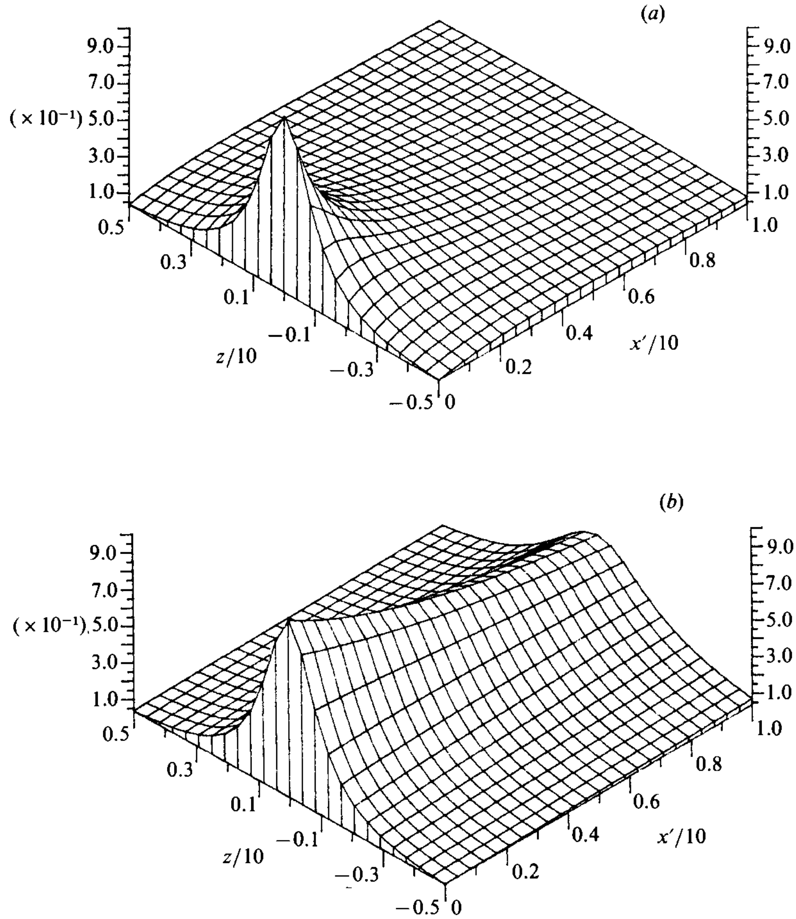

FIGURE 1. (a) Bottom protrusion $A^{-1}f(x',z) = (1+c_0 x)/\{(1+c_0 x')^2+z^2\}$, $c_0 = \cos\pi/30$. (b) Bottom protrusion $A^{-1}f(x',z) = (1+c_6 x')/\{(1+c_6 x')^2+z^2\}$, $c_6 = \cos 13\pi/30$.

Yih for the $n = 0$ case, as the protrusion becomes less peaked and broader in extent in the longshore direction the amplitude of the longest edge-wave mode ultimately diminishes.

In order to illustrate more clearly our results and to assess the relative importance of each of the $n+1$ edge-wave modes we present some representative curves derived from (6.5) and (6.7). In what follows we shall restrict our computations to the case $a = 1$ and a bottom slope of $\alpha = \frac{1}{30}\pi$. We shall also choose $ga/U^2 = 10$.

In figure 1(a, b) we sketch $A^{-1}f(x',z)$ for $p = 0$ and $n = 0, 6$. It can be seen that the effect of increasing n is to extend the influence of the protrusion seawards. Figure 2(a, b) is as figure 1 but with $p = 2$, and we see that the protrusion now has a zero in the longshore directions. Again the effect of increasing n is to extend the protrusion seawards.

Figure 3 shows the variation of $B_r(x)$ with x as computed from (6.7) for the case of $p = 0$ and $n = 6$ as in figure 1(b). Only the first three edge-wave modes are sketched since the rest are negligible in amplitude. The lowest mode is clearly dominant with the subsequent modes rapidly diminishing as r increases.

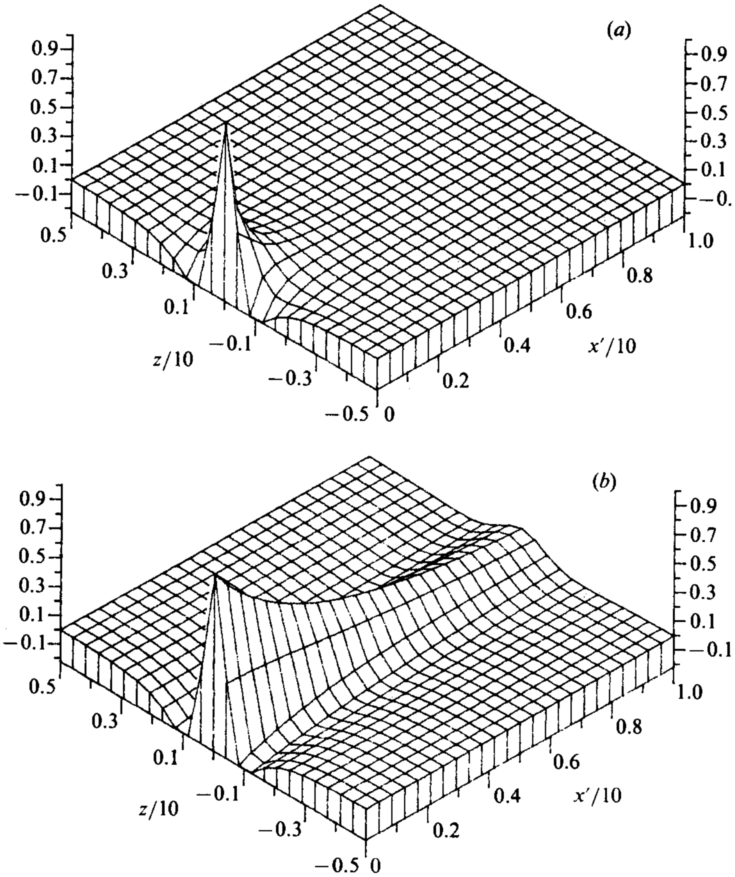

FIGURE 2. (*a*) Bottom protrusion $A^{-1}f(x',z) = (1+c_0x')\{(1+c_0x')^2-3z^2\}/\{(1+c_0x')^2+z^2\}^3$, $c_0 = \cos\pi/30$. (*b*) Bottom protrusion $A^{-1}f(x',z) = (1+c_6x')\{(1+c_6x')^2-3z^2\}/\{(1+c_6x')^2+z^2\}^3$, $c_6 = \cos 13\pi/30$.

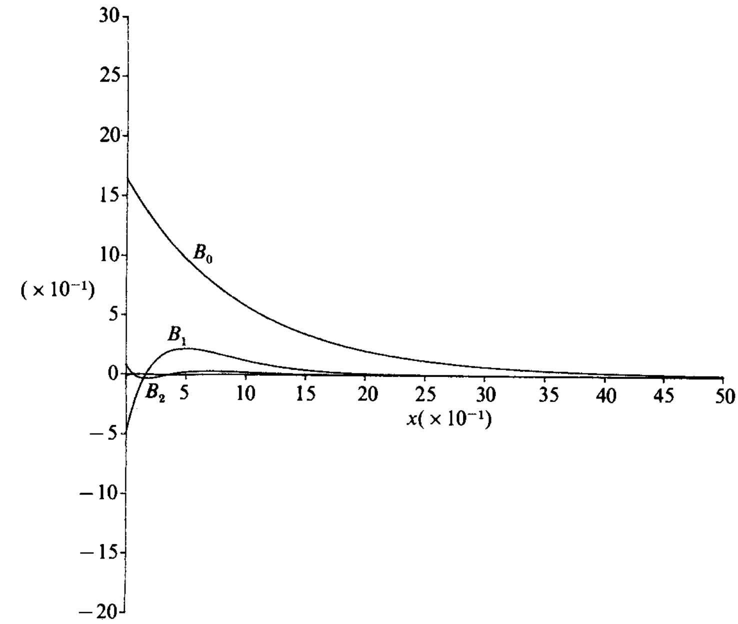

FIGURE 3. Amplitude of edge-wave modes B_r for bottom protrusion of figure 1(*b*), from (6.6), (6.7).

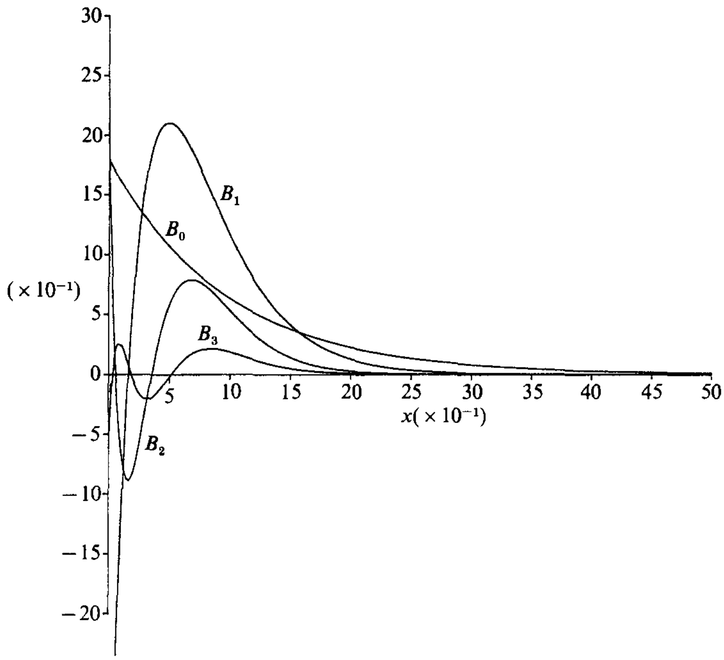

FIGURE 4. Amplitude of edge-wave modes B_r for bottom protrusion of figure 2(*b*), from (6.6), (6.7).

In contrast figure 4 shows the variation of $B_r(x)$ with x for the case $p = 2$ and $n = 6$ as in figure 2(*b*). Here both $B_1(x)$ and $B_2(x)$ exceed $B_0(x)$ in magnitude at $x = 0$, taking values -4.64 and 2.11 there respectively. In fact $B_1(x)$ dominates $B_0(x)$ for a wide range of values of x.

7. Conclusion

It has been shown how Ursell-type standing edge waves of wavelength $2\pi U^2/g \sin(2r+1)\alpha$ can be created by a uniform longshore current U passing over a localized protrusion on an otherwise uniform sloping beach of slope α. The protrusion can be of fairly general form but the rate at which the amplitude of the protrusion decays in the seaward compared to the longshore direction is dictated by the choice of an integer n satisfying $2(n+1)\alpha < \frac{1}{2}\pi$. Then the number of edge-wave modes created is $n+1$. A more restricted bottom profile, obtained by choosing $n = 0$ was considered by Yih (1984) who showed how a single Stokes standing edge-wave could be produced.

Of particular interest was the fact that for a particular shaped protrusion the second edge-wave mode could dominate the first in amplitude, both at the shoreline and at certain distances offshore. There appears to be no reason why higher edge-wave modes should not dominate lower modes for other bottom shapes also. The author is grateful to a referee for pointing out that this possibility is confirmed by the observations of Holman & Bowen (1984) of standing edge waves of mode number in the range 3–7 on a sloping beach, although it is not suggested that the mechanisms in the two cases are the same.

It has also been shown how a localized moving longshore surface-pressure distribution over a uniform sloping beach can also give rise to Ursell-type standing edge waves. The analysis is similar to the longshore-current case and no detailed results are presented. Again the general form of the pressure distribution involves an integer n which must satisfy $(2n+1)\alpha < \frac{1}{2}\pi$. Aside from the technical requirements of convergence of the solution it is not clear whether there is any physical significance in the different restrictions on n in the two cases. The transient development of edge waves is also briefly considered using the same spatial variation (3.6).

The major shortcoming of the present work is the restriction of the forcing mechanism to forms such as (3.6) and (4.5) although certain flexibility is available through the choice of $b(k)$. The determination of the edge waves created by a longshore current over an *arbitrary* bottom protrusion on a sloping beach, or by an *arbitrary* localized longshore surface-pressure distribution, by linear theory, remains an elusive goal.

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